Symplectic inverse spectral theory for pseudodifferential operators

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Abstract

We prove, under some generic assumptions, that the semiclassical spectrum modulo $\mathcal{O}(\hbar^2)$ of a one dimensional pseudodifferential operator completely determines the symplectic geometry of the underlying classical system. In particular, the spectrum determines the hamiltonian dynamics of the principal symbol.

1 Introduction

In this article I would like to advocate an inverse spectral theory for pseudodifferential operators. What does this means ? One of the most famous inverse spectral problems, made fashionable by Kac's very entertaining article [11], with a mind-catching title "Can one hear the shape of a drum ?"(1), was about the Laplace operator on a bounded domain $\Omega \subset \mathbb{R}^n$. Frequencies ν solutions to the eigenvalue problem

$$\frac{1}{2}\Delta u = \nu^2 u, \qquad u = 0 \text{ on } \partial\Omega$$

may be viewed as harmonics that can be heard when the interior of the "membrane" Ω vibrates freely. The question was whether the knowledge of *all* frequencies completely determines Ω (up to isometry, of course). As Kac mentioned, this question appears naturally in the context of quantum mechanics, for a particle trapped in a hard potential well. An important observation in this paper was the relevance of the *Weyl law*, which let us

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(1) Kac attributes the problem to Bochner and the title to Bers.

find the *volume* (or area when $n = 2$) of Ω from the *asymptotic behaviour of large eigenvalues*.

Counterexamples are now known : there are non-isometric shapes in $\mathbb{R}^2$ that produce different frequencies [9]. Nonetheless, this fact should not let us think that the problem has become obsolete. As far as I know, the seemingly simple case of a convex, bounded domain $\Omega \in \mathbb{R}^2$ with analytic boundary is still open (although by the time this article is published, it might very well have been settled by Zelditch, see [22, 21]).

Understanding this problem requires putting it in a wider perspective. A natural variant of Kac's problem is whether the spectrum of the Laplace operator Δ_g on a compact riemannian manifold (M, g) determines the metric g. Although, here again, counterexamples have been known for a long time [14], our understanding remains relatively poor. Recent works by Zelditch and Guillemin suggest that microlocal tools are quite relevant for all these questions. This, in turn, is a hint that more general operators than the Laplacian could be dealt with similarly.

From a quantum mechanical viewpoint, Kac's situation is quite extreme. A more natural setting would involve a particle 'trapped' by a smooth potential well. No more boundary problems, but instead a Schrödinger operator on $\mathbb{R}^n$

$$P = -\frac{\hbar^2}{2}\Delta + V(x).$$

Of course now the *potential function* V should be recovered from the spectrum of P. This inverse spectral problem has been studied a lot, but only very recently have microlocal tools similar to those used by Guillemin and Zelditch been applied to it [10, 5].

Here, I would like to shift again the initial problem one step further away. Instead of the Laplacian, or the Schrödinger operator, why not consider *any* (elliptic) differential operator, or even, since we're at it, any *pseudodifferential* operator ? Of course, since there is no domain Ω anymore, no potential function V, the sensible question is *what* should we try to recover from the spectrum ?

The inverse spectral problems I've mentioned here can all be understood as *semiclassical limits*. From a quantum object, the spectrum, one wants to recover classical observables such as the metric g, or the potential V. These quantities, in turn, fully determine the *classical dynamics* of the system. For general pseudodifferential operators, semiclassical analysis still shows the strong relationship between the classical dynamics and the quantum spectrum, so I believe that the most natural "object" that we should try and recover from the spectrum is precisely this classical dynamics. This,

precisely, amounts to determining the *principal symbol* of the operator. In fact, if we keep in mind Weyl's asymptotics, this sounds fairly natural, for it is well known that Weyl's asymptotics extend to arbitrary pseudodifferential operators, provided that we compute phase space volumes defined by energy ranges given by the principal symbol [15].

As in the riemannian case, one should take into account a symmetry group acting on the classical data. For general pseudodifferential operators, there's only one available : the group of symplectomorphisms, acting on the phase space M. This is a much bigger group than the group of riemannian isometries, in accordance with the fact that the space of principal symbols $C^\infty(M)$ is much bigger than the space of riemannian metrics, or potential functions.

2 The setting

Since we aim at recovering the classical dynamics from the spectrum, we are going to work in the setting of semiclassical pseudodifferential operators, which we recall here. Throughout this work, we only consider the *one-dimensional* theory. It would be very interesting to have higher dimensional results, but it is not expected that such precise results would persist. However, a reasonable challenge would be to undertake a similar study for the *completely integrable* case.

The classes $\Psi^d(m)$ of semiclassical pseudodifferential operators we use are standard. Let $M = T^*\mathbb{R} = \mathbb{R}^2_{(x,\xi)}$. Let d and m be real numbers. Let $S^d(m)$ be the set of all families $(p(\cdot;\hbar))_{\hbar\in(0,1]}$ of functions in $C^\infty(M)$ such that

$$\forall \alpha \in \mathbb{N}^2, \quad \left|\partial^\alpha_{(x,\xi)} p(x,\xi;\hbar)\right| \leqslant C_\alpha \hbar^d (1+|x|^2+|\xi|^2)^{\frac{m}{2}}, \tag{1}$$

for some constant $C_\alpha > 0$, uniformly in $\hbar$. Then $\Psi^d(m)$ is the set of all (unbounded) linear operators P on $L^2(\mathbb{R})$ that are $\hbar$-Weyl quantisations of symbols $p \in S^d(m)$:

$$(Pu)(x) = (Op^w_\hbar(p)u)(x) = \frac{1}{2\pi\hbar}\int_{\mathbb{R}^2} e^{\frac{i}{\hbar}\langle x-y,\xi\rangle} p(\tfrac{x+y}{2},\xi;\hbar)u(y)\,|dyd\xi|\,.$$

The number d in (1) is called the $\hbar$-order of the operator. Unless specified, it will always be zero here. In this work all symbols are assumed to admit a "classical" asymptotic expansions in integral powers of $\hbar$ (that is to say, in the ladder $(S^d(m))_{d\in\mathbb{Z},d\geqslant d_0}$ for some $d_0 \in \mathbb{Z}$). The leading term in this expansion is called the principal symbol of the operator.

Thus, the Schrödinger operator $P = -\frac{\hbar^2}{2}\Delta + V$ on $\mathbb{R}$ is a good candidate, of $\hbar$-order zero, whenever V has at most a polynomial growth.

We use in this article the standard properties of such pseudodifferential operators. In particular the composition sends $\Psi^d(m) \times \Psi^{d'}(m')$ to $\Psi^{d+d'}(m+m')$. Moreover all $P \in \Psi^0(0)$ are bounded: $L^2(\mathbb{R}) \to L^2(\mathbb{R})$, uniformly for $0 < \hbar \leqslant 1$.

An operator $P \in \Psi(m)$ is said to be *elliptic at infinity* if there exists a constant $C > 0$ such that the principal symbol p satisfies

$$|p(x,\xi)| \geqslant \frac{1}{C}(|x|^2 + |\xi|^2)^{m/2}$$

for $|x|^2 + |\xi|^2 \geqslant C$.

If P has a real-valued Weyl symbol, then it is a symmetric operator on L^2 with domain $C_0^\infty(\mathbb{R})$. If furthermore the principal symbol is elliptic at infinity, then P is essentially selfadjoint (see for instance [7, proposition 8.5]).

Finally, when $P \in \Psi^0(m)$ is selfadjoint and elliptic at infinity, then for any $f \in C_0^\infty(\mathbb{R})$, the operator $f(P)$ defined by functional calculus satisfies $f(P) \in \cap_{k\in\mathbb{N}}(\Psi^0(-km))$. See for instance [7] or [16] for details.

The advantage of the semiclassical theory is that it allows us to use richer versions of Weyl's asymptotics. Instead of considering the limit of large eigenvalues, we fix a bounded *spectral window* $I = [E_0, E_1] \subset \mathbb{R}$ and study the asymptotics of all eigenvalues in I, as $\hbar \to 0$.

Definition 2.1 *We say that Assumption $\mathcal{A}(P, \mathcal{J}, I)$ holds whenever*

1. *P is a selfadjoint pseudodifferential operator in $\Psi^0(m)$ with principal symbol p, elliptic at infinity.*
2. *$\mathcal{J} \subset [0,1]$ is an infinite subset with zero as an accumulation point.*
3. *There exists a neighbourhood J of I such that $p^{-1}(J)$ is compact in M.*

If Assumption $\mathcal{A}(P, \mathcal{J}, I)$ holds, we denote by $\Sigma_\hbar(P, I)$ the spectrum of $P = P(\hbar)$ in I (including multiplicities). We denote by $\Sigma(P, \mathcal{J}, I)$ the family of all $\{\Sigma_\hbar(P, I); \quad \hbar \in \mathcal{J}\}$. It is well known that $\Sigma_\hbar(P, I)$ is discrete for $\hbar$ small enough (see eg. [16, Théorème 3.13], in a slightly different setup). Notice that when $m > 0$, the properness condition 3. is always satisfied.

Proposition 2.2 *Let P be a selfadjoint pseudodifferential operator in $\Psi(m)$, with principal symbol p, elliptic at infinity. Let $J \subset \mathbb{R}$ be a closed interval*

such that $p^{-1}(J)$ is compact. Then for any open interval $I \subset J$ there exists $\hbar_0 > 0$ such that the spectrum of P in I is discrete for $\hbar \leqslant \hbar_0$.

Proof. The case $m > 0$ is probably the most standard. We recall it quickly.

Case $m > 0$**.** — Let $\chi \in C_0^\infty(J)$ be equal to 1 on I. Then by pseudodifferential functional calculus, $f(P)$ is compact for $\hbar$ small enough. Therefore, denoting by Π_I the spectral projector on I, we have that $\Pi_I = \Pi_I f(P)$ is compact. This implies that Π_I has finite rank : the spectrum in I is discrete.

Case $m \leqslant 0$. — First we show that one can replace J by an unbounded interval containing I. Thus assume J is compact. For notational convenience we let $J = [-1, 0]$. For any $\alpha \in]-1, 0[$, Sard's theorem ensures the existence of a regular value $\lambda \in]\alpha, 0[$ for p. Then $\mathcal{C} := p^{-1}(\lambda)$ is a compact 1-dimensional submanifold of $\mathbb{R}^2$: it is a finite union of circles. Let Ω be the unbounded component of $\mathbb{R}^2 \setminus \mathcal{C}$. Suppose first that $p_{\restriction\Omega} > \lambda$. Since the differential of p does not vanish on $\mathcal{C}$, $p < \lambda$ in all the bounded components of $\mathbb{R}^2 \setminus \mathcal{C}$. Therefore $p^{-1}(\,]-\infty, \lambda])$ is compact and one may replace J by $]-\infty, \lambda]$. Now if on the contrary $p_{\restriction\Omega} < \lambda$ on the unbounded components, we have to apply the same argument for $\lambda' \in]-1, \alpha[$. Because $\lambda' < \lambda$, the new bounded components contain the old one, and therefore consist of the points where $p > \lambda'$. Then $p^{-1}([\lambda', +\infty[\,)$ is compact, and one may replace J by $[\lambda', +\infty[$.

For the rest of the proof, we may suppose that $J =]-\infty, 0]$. Let $-\epsilon \in J \setminus I$, close to the origin. Let $\chi \in C^\infty(\mathbb{R})$ be such that

$$\begin{cases} \chi(x) = -\epsilon/2 & \text{for } x \in]-\infty, -\epsilon] \\ \chi(x) = x & \text{for } x \geqslant -\epsilon/3 \\ \chi(x) \geqslant -\epsilon/2 & \text{everywhere.} \end{cases}$$

Let $p_\hbar$ be the Weyl symbol of P and define $\tilde{p}_\hbar = \chi \circ p_\hbar$. Then $\tilde{p}_\hbar$ is a symbol in $\Psi(m)$, with $\tilde{p}_\hbar \geqslant -\epsilon/2$ on $\mathbb{R}^2$ and $\tilde{p}_\hbar = p_\hbar$ outside the set $p_\hbar^{-1}(\,]-\infty, -\epsilon/3])$. Because $m \leqslant 0$, the set $p_\hbar^{-1}(\,]-\infty, -\epsilon/3])$ is included in $p^{-1}(J)$ for $\hbar$ small enough and hence must be compact as well. Denote by $\tilde{P}$ the Weyl quantisation of $\tilde{p}$. Then for $\hbar$ small enough $(\tilde{P} - \lambda)$ is invertible for all $\lambda \in I$. We can write

$$P - \lambda = (\tilde{P} - \lambda)\left(\mathrm{Id} + (\tilde{P} - \lambda)^{-1}(P - \tilde{P})\right).$$

Since $P - \tilde{P}$ is the Weyl quantisation of a compactly supported symbol (with support in a compact independent of $\hbar$), it is of the trace class. By analytic

Fredholm theory, we may take the determinant of $(\mathrm{Id}+(\tilde{P}-\lambda)^{-1}(P-\tilde{P}))$ and conclude that the spectrum of P consists of the zeroes (with multiplicities) of a non-vanishing holomorphic function and hence is discrete. $\square$

The goal of this article is to recover the dynamics of the hamiltonian p in the region $p^{-1}(I)$ for any operator P for which Assumption $\mathcal{A}(P, \mathcal{J}, I)$ holds, for some subset $\mathcal{J} \subset [0, 1]$. Of course if we can do it for an arbitrary compact interval $I \subset \mathbb{R}$, we recover the full dynamics of p.

It turns out that, under some genericity conditions, these inverse spectral problems are fairly easy, compared to the general multi-dimensional problems alluded to in the introduction, in the sense that they only require a few terms in the asymptotics of the spectrum. Having this in mind, for $\alpha \in \mathbb{R}$ we denote by $\Sigma(P, \mathcal{J}, I) + \mathcal{O}(\hbar^\alpha)$ the equivalence class of all $\Sigma_\hbar(P, I)$ modulo $\hbar^\alpha$. Our main result is Theorem 5.2, but we also state several intermediate results that require weaker hypothesis. An informal statement of Theorem 5.2 is as follows.

Theorem 2.3 (Theorem 5.2) *Let Assumption $\mathcal{A}(P, \mathcal{J}, I)$ hold, and denote $M = p^{-1}(I)$. Suppose that $p_{\restriction M}$ is a Morse function. Assume that the graphs of the periods of all trajectories of the hamiltonian flow defined by $p_{\restriction M}$, as functions of the energy, intersect generically.*

Then the knowledge of $\Sigma(P, \mathcal{J}, I) + \mathcal{O}(\hbar^2)$ determines the dynamics of the hamiltonian system $p_{\restriction M}$.

In fact, we determine completely the Hamiltonian p up to symplectic equivalence. Perhaps the most difficult step, for which Weyl's asymptotics are not enough, is the seemingly simple question to count the number of connected components of $p^{-1}(E)$, for a regular energy $E \in I$ (Theorem 4.2).

Although we state everything for pseudodifferential operators defined on on $\mathbb{R}$, it is most probable that all results extend to the case of pseudodifferential operators defined on a one-dimensional compact manifold equipped with a smooth density, and to the case of Toeplitz operators on two-dimensional symplectic manifolds.

The plan of the paper follows a fairly logical progression. Since we always work modulo symplectomorphisms, it is not reasonable to look for a formula that would give the principal symbol p. Instead we will try to recover as many *symplectic invariants* as possible from the spectrum so that, given two spectra, we should be able to tell whether they come from isomorphic systems.

Thus, the geometric object under study is a proper map $p : M \to \mathbb{R}$, where M is a symplectic 2-manifold. The simplest symplectic invariants of this map are in fact topological invariants, and are dealt with in Sections 3 and 4. Indeed, it follows from the action-angle theorem that as soon as $E \in \mathbb{R}$ is a regular value of p, then the fibres of p consist of a finite number of closed loops, each one diffeomorphic to a circle. Therefore, we need to be able to detect

1. Whether an energy $E \in \mathbb{R}$ is a regular or critical value of p; this is done in Section 3 (Theorem 3.1).
2. When E is a regular value, the number of connected components of the fibre $p^{-1}(E)$; Section 4.1 discusses this point (Theorem 4.2).

Putting these results together we are able to recover the topological type of the singular fibration (Theorem 4.5). Then in Section 5, relying on the classification result of Dufour-Molino-Toulet [8, 17] (and some additional argument) we finally manage to recover the symplectic geometry of the system (Theorem 5.2).

3 Singularities

In order to detect whether a given energy $E_0 \in \mathbb{R}$ is a critical value of p or not, it is enough to know the spectrum of P in a small ball around E_0, at least under some nondegeneracy conditions.

Recall that a function $f : M \to \mathbb{R}$ is said to have a nondegenerate critical point $m \in M$ when $df(m) = 0$ and the Hessian $f''(m)$ is a nondegenerate quadratic form. Since M has dimension 2, there are only two cases :

1. **Elliptic case :** there are local symplectic coordinates (x, ξ) in T_mM such that $f''(m)(x, \xi) = C(x^2 + \xi^2)$, for some constant $C \neq 0$.
2. **Hyperbolic case :** there are local symplectic coordinates (x, ξ) in T_mM such that $f''(m)(x, \xi) = Cx\xi$, for some constant $C \neq 0$.

We refer to each of these two cases as the *type* of the singularity m.

Theorem 3.1 *Let I be an interval containing E_0 in its interior, and let Assumption $\mathcal{A}(P, \mathcal{J}, I)$ hold. Assume also that p has only nondegenerate critical values in I, and that any two critical points with the same singularity type cannot have the same image by p. Then from the knowledge of $\Sigma(P, \mathcal{J}, I) + \mathcal{O}(\hbar^2)$ one can infer*

1. *whether E_0 is a critical value of p or not;*
2. *in case E_0 is a critical value, the type of the singularity.*

This theorem is a corollary of the following proposition, where we consider the *density of states* in small regions around E_0. We could, equivalently, invoke Weyl's asymptotics.

Proposition 3.2 *Let $\gamma \in (0,1)$ and, for $E \in I$,*

$$\rho_\hbar(E) = \hbar^{1-\gamma} \#(\Sigma_\hbar(P, B(E, \hbar^\gamma))).$$

Then for any $E \in I$, the limit $\rho(E) = \lim_{\hbar \to 0} \rho_\hbar(E)$ exists (in $[0, +\infty]$), and

1. *if E is a regular value of p, then ρ is smooth at E;*
2. *if E is an elliptic critical value of p, then ρ is discontinuous;*
3. *if E is a hyperbolic critical value of p, then $\rho(E) = +\infty$.*

Proof. In case E is a regular value, the result follows directly from Weyl's asymptotics, which in turn can be derived from a semiclassical trace formula as in [6], or from the semiclassical Bohr-Sommerfeld rules as in [18]. Let us recall the Bohr-Sommerfeld approach. There exists an $\epsilon > 0$ such that the eigenvalues of P inside $[E - \epsilon, E + \epsilon]$ modulo $O(\hbar^\infty)$ are the union (with multiplicities) of a finite number of spectra σ_k, $k = 1, \dots, N$, where N is the number of connected components of $p^{-1}(E)$, and each σ_k is determined by quasimodes microlocalised on the corresponding component. Precisely, the elements of σ_k are given by the solutions λ to the equation

$$g^{(k)}(\lambda; \hbar) \in 2\pi\hbar\mathbb{Z}, \tag{2}$$

where the function $g^{(k)}$ admits an asymptotic expansion of the form

$$g^{(k)}(\lambda; \hbar) \simeq g_0^{(k)}(\lambda) + \hbar g_1^{(k)}(\lambda) + \hbar^2 g_2^{(k)}(\lambda) + \cdots \tag{3}$$

with smooth coefficients g_j. Moreover, if we denote by $\mathcal{C}_k(\lambda)$ the k-ieth connected component of $p^{-1}(\lambda)$, in such a way that the family $(\mathcal{C}_k(\lambda))$ is smooth in the variable λ, then $g_0^{(k)}$ is the *action integral* :

$$g_0^{(k)}(\lambda) = \int_{\mathcal{C}_k(\lambda)} \xi dx. \tag{4}$$

From (2) is follows that, for $\hbar$ small enough

$$\# (\sigma_k \cap B(E, \epsilon)) = (2\pi\hbar)^{-1} \left| g^{(k)}(E + \epsilon; \hbar) - g^{(k)}(E - \epsilon; \hbar) \right| + \delta,$$

where the $\delta \in [-1, 1]$ is here to take care of the appropriate integer part of the right-hand-side. Hence

$$\#\left(\sigma_k \cap B(E, \epsilon)\right) = (2\pi\hbar)^{-1} \left| 2\epsilon \frac{\partial g_0^{(k)}(E)}{\partial E} + \mathcal{O}(\epsilon^2) + \mathcal{O}(\hbar) \right| + \delta.$$

With $\epsilon = \hbar^\gamma$, this gives

$$\#\left(\sigma_k \cap B(E, \hbar^\gamma)\right) = \frac{\hbar^{\gamma-1}}{\pi} \left| \frac{\partial g_0^{(k)}(E)}{\partial E} \right| + \mathcal{O}(\hbar^{2\gamma-1}) + \mathcal{O}(1).$$

Summing up all contributions for $k = 1, \ldots, N$, we get the first claim of the theorem, with

$$\rho(E) = \frac{1}{\pi} \left| \frac{\partial g_0^{(k)}(E)}{\partial E} \right|.$$

The second claim can be proved in a similar way, using Bohr-Sommerfeld rules for elliptic singularities [20]. For our purposes, a Birkhoff normal form as in [2] would even be enough, since we deal with energy intervals of size $\mathcal{O}(\hbar^\gamma)$. Here again there exists an $\epsilon > 0$ such that the eigenvalues of P inside $[E - \epsilon, E + \epsilon]$ modulo $O(\hbar^\infty)$ are the union (with multiplicities) of a finite number of spectra σ_k corresponding to the various connected components of $p^{-1}(E)$. The difference is that not all components need have critical points. In fact by assumption only one component may have an elliptic critical point. Let us call σ_k the corresponding spectrum, and $\mathcal{C}_k(\lambda)$ the corresponding family of connected components. Since an elliptic critical point is a local extremum for p, the sets $\mathcal{C}_k(\lambda)$ are empty for all λ in one of the halves of the interval $[E - \epsilon, E + \epsilon]$. Without loss of generality, one can assume that $\mathcal{C}_k(\lambda) = \emptyset$, $\forall \lambda \in [E - \epsilon, E[$. Then $\mathcal{C}_k(E)$ is just a point, while $\mathcal{C}_k(\lambda)$ is a circle for all $\lambda \in]E, E + \epsilon]$.

The Bohr-Sommerfeld rules for elliptic singularities say that the elements of σ_k are the solutions λ to an equation of the form

$$e^{(k)}(\lambda; \hbar) \in 2\pi\hbar\mathbb{N}, \tag{5}$$

where the function $e^{(k)}$ admits an asymptotic expansion exactly as $g^{(k)}$ above (3). What's more, it is equally true that the principal term is an action integral :

$$e^{(k)}(E) = 0, \qquad e_0^{(k)}(\lambda) = \int_{\mathcal{C}_k(\lambda)} \xi dx, \quad \forall \lambda \in [E, E + \epsilon].$$

$\square$

Calculating along the same lines as above, we find, for the quantity

$$\rho_\hbar^{(k)}(\lambda) := \hbar^{1-\gamma}\#(\sigma_k \cap B(\lambda, \hbar^\gamma))),$$

the following limits :

1. when $\lambda \in [E-\epsilon, E[$, $\lim_{\hbar\to 0} \rho_\hbar^{(k)}(\lambda) = 0$;
2. when $\lambda \in]E, E+\epsilon, E]$, $\lim_{\hbar\to 0} \rho_\hbar^{(k)}(\lambda) = \frac{1}{\pi}\left|\frac{\partial e_0^{(k)}(\lambda)}{\partial \lambda}\right|$;
3. $\lim_{\hbar\to 0} \rho_\hbar^{(k)}(E) = \frac{1}{2\pi}\left|\frac{\partial e_0^{(k)}(E)}{\partial E}\right|$;

Finally, let E be a hyperbolic critical value for p. Weyl asymptotics for such a situation have been worked out in [1], and the singular Bohr-Sommerfeld rules have been established in [3]. Using the latter result it can be proven as in [12] that the number of semiclassical eigenvalues generated by a hyperbolic fixed point, in a neighbourhood of size $\epsilon = \hbar^\gamma$ of the critical value, is of order $\epsilon\,|\ln\hbar|\,/\hbar$. Therefore, since there may be only one hyperbolic point in $p^{-1}(0)$, it follows from this estimate and the results we just proved above for the regular and the elliptic case that

$$\rho_\hbar(E) \geqslant C\,|\ln\hbar|\,,$$

for some constant $C > 0$. This gives $\rho(E) = +\infty$.

Remark 3.3 It is probable that the nondegeneracy condition can be avoided. It is known quite generally that Weyl asymptotics hold for critical energies [23]. Thus, in all case, we recover the action integral as the integrated density of states. It would remain to show that the behaviour of the action integral completely determines the singularities of p. This is easy in the Schrödinger case $p = \xi^2 + V(x)$. $\triangle$

4 Topology

As we already mentioned above, once the singular fibres of p have been excluded, the topology is easy to understand. The map p become a locally trivial fibration whose fibres are disjoint unions of circles.

Thus, if E_0 is a regular value of p, the semiglobal problem around E_0 just amounts to counting the number of connected components of $p^{-1}(E_0)$.

The topology of singular fibres strongly depends on the type of singularity. Under the nondegeneracy assumption, the topology of the singular foliation in a neighbourhood of a singular fibre is essentially determined by the type of the singularity, and thus by Theorem 3.1.

4.1 Connected components

Let I be a compact interval of regular values of p. As above, we denote by $\mathcal{C}_k(\lambda)$, for $k = 1, \dots, N$ and $\lambda \in I$ the smooth families of connected components of $p^{-1}(\lambda)$. Each $\mathcal{C}_k(\lambda)$ is globally invariant by the hamiltonian flow generated by p. Thus, this flow is periodic on $\mathcal{C}_k(\lambda)$. Let $|\tau_k(\lambda)| \neq 0$ be its primitive period (the sign is determined by the formula below). It follows from the action-angle theorem that τ_k is a smooth function of λ. In fact it is well know that the period is the derivative of the action, and we have already met this quantity in the proof of Proposition 3.2. Using the action integral (4), we get

$$\tau_k(\lambda) = \frac{\partial g_0^{(k)}(\lambda)}{\partial \lambda}.$$

Notice again that τ_k never vanishes on I.

Definition 4.1 *We say that a point $(\lambda, t) \in (I \times \mathbb{R}^*)$ is* ***resonant*** *whenever there exist (k, j) and (k', j') in $\{1, \dots, N\} \times \mathbb{Z}^*$, with $k \neq k'$, such that*

$$j\tau_k(\lambda) = j'\tau_{k'}(\lambda) = -t.$$

Theorem 4.2 *Let I be an interval of regular values of p, and let Assumption $\mathcal{A}(P, \mathcal{J}, I)$ hold. Assume also that the set of resonant points in $I \times \mathbb{R}$ is discrete. Then the number N of connected components of $p^{-1}(\lambda)$, $\lambda \in I$, is determined by spectrum $\Sigma(P, \mathcal{J}, I) + \mathcal{O}(\hbar^2)$.*

Before proving the theorem, let us just remark that the leading term of Weyl's asymptotics is not sharp enough for this. Indeed, it only gives the density ρ (Proposition 3.2) :

$$\rho(\lambda) = \frac{1}{\pi} \sum_{k=1}^{N} |\tau_k(\lambda)| \,. \tag{6}$$

From this one cannot distinguish, for example, one component with period τ from two components with periods $|\tau_1| + |\tau_2| = |\tau|$.

Remark also that the condition on resonant points is not adapted to systems with symmetries. For instance, a Schrödinger operator with a symmetric double well has two components with equal periods.

Proof *of Theorem 4.2***.** We introduce the period lattice $\mathcal{L}_k(I)$:

$$\begin{aligned}\mathcal{L}_k(I) :&= \{(\lambda, t) \in I \times \mathbb{R}; \quad \exp(t\mathcal{X}_p) \text{ is periodic on } \mathcal{C}_k(\lambda)\} \\ &= \{(\lambda, j\tau_k(\lambda)); \quad \lambda \in I, j \in \mathbb{Z}\}\end{aligned}$$

and $\mathcal{L}(I) = \bigcup_{k=1}^{N} \mathcal{L}_k(I)$. The set $\mathcal{L}(I)$ is a union of smooth graphs that may intersect. The intersection points for $t \neq 0$ are precisely the resonant points.

In order to prove the theorem, we split the argument into two steps. The first one is to prove that $\Sigma(P, \mathcal{J}, I) + \mathcal{O}(\hbar^2)$ determines $\mathcal{L}(I)$. The second step consists in showing why the knowledge of $\mathcal{L}(I)$ — and the assumption on the set of resonant points — allows us to count the number N of connected components.

Step 1. Coming back to the Bohr-Sommerfeld rules discussed in the proof of Proposition 3.2, we recall that the spectrum of P modulo $O(\hbar^\infty)$ is the superposition of the spectra σ_k generated by $\mathcal{C}_k$, for $k = 2, \ldots, N$. For each k, σ_k has a periodic structure that makes it close to an arithmetic progression. Thus, a simple and naive idea to distinguish between the different periodic structures is to perform a *frequency analysis*, via a Fourier transform. Because we have at our disposal only a truncated sequence of eigenvalues (those that belong to I), we need to introduce a cut-off. Let $I' \Subset I$ and let $\chi \in C^\infty(\mathbb{R})$ have compact support in the interior of I and be equal to 1 on I'. We introduce the spectral measure

$$D_0(\lambda; \hbar) = \sum_{E \in \Sigma_\hbar(P,I)} \chi(E)\delta_E(\lambda),$$

where δ_E is the Dirac distribution at E. The quantity we want to investigate is its Fourier transform. Since the mean spacing between consecutive eigenvalues is of order $\hbar$, we use a corresponding scale for the time variable t, and thus introduce

$$Z(t; \hbar) = \sum_{E \in \Sigma_\hbar(P,I)} \chi(E) e^{-itE/\hbar}.$$

The function Z is called the *partition function*. In fact, the idea we've just described is very well known in the semiclassical context, and is part of

the general formalism of *trace formulæ*. We can consider the Schrödinger group $U(t;\hbar) = \exp(-itP/\hbar)$, and then $Z(t;\hbar) = \mathrm{Trace}(\chi(P)U(t;\hbar))$. It is well known that $\chi(P)U(t;\hbar)$ is a Fourier Integral Operator, whose canonical transformation is the classical flow of p. Moreover, its trace is a lagrangian (or WKB) distribution associated with the lagrangian manifold of periods

$$\Lambda_p = \{(E,\tau) \in \mathbb{R}^2; \quad \exists z \in p^{-1}(E), \exp(\tau\mathcal{X}_p)(z) = z\} = \mathcal{L}(I').$$

Such a result would almost finish the proof of Step 1. In fact, this statement exists in many versions, depending on various possible situations and hypothesis. For this reason we are not using it here as is, but instead resort once again to the Bohr-Sommerfeld rules, which is arguably the easiest way to go.

We can split the partition function as

$$Z(t;\hbar) = \sum_{k=1}^{N} \sum_{E\in\sigma_k} \chi(E)e^{-itE/\hbar}.$$

Then from (2) one can introduce $c \mapsto f^{(k)}(c;\hbar)$ as the inverse of $\lambda \mapsto g^{(k)}(\lambda;\hbar)$, which exists for $\hbar$ small enough, and write

$$Z(t;\hbar) = \sum_{j\in\mathbb{Z}} \varphi_t(2\pi\hbar j;\hbar) \tag{7}$$

(which, as before, is a finite sum) with

$$\varphi_t(c;\hbar) := \sum_{k=1}^{N} \chi(f^{(k)}(c;\hbar))e^{-itf^{(k)}(c;\hbar)/\hbar}. \tag{8}$$

Note that $\varphi_t(\cdot;\hbar) \in C_0^\infty(\mathbb{R})$. By the Poisson summation formula,

$$Z(t;\hbar) = \frac{1}{2\pi\hbar} \sum_{j\in\mathbb{Z}} \hat{\varphi}_t(j/\hbar) \tag{9}$$

(which, contrary to (7), is a truly infinite sum) with

$$\hat{\varphi}_t(j/\hbar) = \int_{\mathbb{R}} e^{-icj/\hbar}\varphi_t(c)dc = \sum_{k=1}^{N} Z_k(t;j,\hbar) \tag{10}$$

and

$$Z_k(t;j,\hbar) = \int e^{-i\hbar^{-1}(cj+tf^{(k)}(c;\hbar))}\chi(f^{(k)}(c;\hbar))dc.$$

The integral Z_k is a compactly supported oscillatory integral, whose phase is stationary when $j + t\frac{\partial f_0^{(k)}}{\partial c} = 0$ or equivalently

$$t = -j\tau_k(\lambda), \qquad \lambda = f^{(k)}(c;\hbar). \tag{11}$$

Moreover, the Hessian of the phase, $Q := t\frac{\partial^2 f_0^{(k)}}{\partial c^2} = t(\frac{\partial \tau_k}{\partial \lambda})^{-1}$ never vanishes for $t \neq 0$. Hence, by the stationary phase expansion, Z_k is a lagrangian distribution whose principal symbol ι_k can be written as a smooth function of λ :

$$\iota_k(\lambda; j, \hbar) = \frac{e^{i\frac{\pi}{4}\operatorname{sign}(Q)}}{|Q|^{1/2}} e^{-ij\hbar^{-1}(g_0^{(k)}(\lambda) - \lambda\tau_k(\lambda))}\chi(\lambda),$$

with $Q = -j\tau_k(\lambda)(\frac{\partial\tau_k}{\partial\lambda})^{-1}$. Since its amplitude vanishes precisely with χ, we can deduce that the semiclassical wave-front of Z_k is (for fixed $j \in \mathbb{Z}^*$)

$$WF_\hbar(Z_k) = \{(\lambda, t) \in \mathbb{R}^2; \quad t = -j\tau_k(\lambda), \chi(\lambda) \neq 0\}.$$

We still need to sum up all $Z_k(t; j, \hbar)$ for $j \in \mathbb{Z}^*$. For this we consider the localisation of Z. Without loss of generality, one can restrict to positive times. Let $t_0 > 0$, $\epsilon > 0$, and let $\rho \in C_0^\infty(B(t_0, \epsilon))$. There is no solution to (11) in the support of ρ for $|j|$ outside the interval

$$I_k(\epsilon) := \left(\frac{t_0 - \epsilon}{\sup_J |\tau_k|}, \frac{t_0 + \epsilon}{\inf_J |\tau_k|}\right).$$

Making explicit the non-stationary phase argument, we can write, for any $\ell \in \mathbb{N}$,

$$Z_k(t; j, \hbar) = \left(\frac{\hbar}{ji}\right)^\ell \int e^{-i\hbar^{-1}(cj + tf_0^{(k)}(c;\hbar))} L^\ell(a(c;\hbar))dc,$$

where L is the linear differential operator defined by

$$(Lu)(c) = \frac{d}{dc}\left(\frac{u(c)}{1 + \frac{t}{j}\frac{\partial f_0^{(k)}}{\partial c}}\right)$$

and $a(\cdot;\hbar) \in C_0^\infty(I)$ admits an asymptotic expansion in non-negative powers of $\hbar$, in the C^∞ topology. Let $b(c) = (1 + \frac{t}{j}\frac{\partial f_0^{(k)}}{\partial c})^{-1}$. Then b is uniformly bounded on I for $|j| > (t_0 + \epsilon)/\inf_J |\tau_k|$, and for any $\ell \in \mathbb{N}^*$, there exists a positive constant C_ℓ, independent of j and $\hbar$, such that $\left|\frac{d^\ell b}{dc^\ell}\right| \leqslant C_\ell / j$. Therefore, there exist constants $\tilde{C}_\ell > 0$ such that

$$\left|L^\ell(a)\right| \leqslant \tilde{C}_\ell,$$

and we get, again when $|j| > (t_0+\epsilon)/\inf_J |\tau_k|$,

$$|\rho(t)Z_k(t;j,\hbar)| \leqslant \tilde{C}_\ell \left(\frac{\hbar}{j}\right)^\ell .$$

Thus, for $\ell \geqslant 2$,

$$\sum_{|j|>\frac{t_0+\epsilon}{\inf_J|\tau_k|}} |\rho(t)Z_k(t;j,\hbar)| \leqslant \tilde{C}_\ell \hbar^\ell .$$

This shows that only a finite (independent of $\hbar$) number of terms contribute to $\rho(t)Z(t;\hbar)$ modulo $O(\hbar^\infty)$. Thus the (non)-stationary phase approximations are jointly valid. Therefore $Z(t;\hbar)$ microlocally vanishes at any point that does not belong to $\mathcal{L}(I)$; this writes

$$WF_\hbar(Z(\cdot;\hbar)) \subset \mathcal{L}(I).$$

More precisely,

$$WF_\hbar(\rho Z(\cdot;\hbar)) \subset \{(\lambda, j\tau_k(\lambda)); \quad \lambda \in I, |j| \in I_k(\epsilon), k=1,\dots,N\}.$$

Moreover, at a non-resonant point $(\lambda, j\tau_k(\lambda))$, no other period $j'\tau_{k'}$ can contribute, and $Z(\cdot;\hbar)$ is a lagrangian distribution with principal symbol equal to $\iota_k(\lambda;j,\hbar)$. Since the set of resonant points is discrete, and $WF_\hbar(Z)$ is closed in T^*I, we must have $WF_\hbar(Z(\cdot;\hbar)) = \mathcal{L}(I)$, which finishes the proof of the first step.

Step 2. We are now left with a simple geometric inverse problem : given the set of periods $\mathcal{L}(I)$, how can one recover the number N of connected components ?

Our strategy is to recover the fundamental periods $|\tau_1|,\dots,|\tau_N|$. First of all, by Weyl's asymptotics (6), one obtains the *a priori* bound $|\tau_k(\lambda)| \leqslant \pi\rho(\lambda)$. Let $R := \max_J \pi\rho$. Then by assumption, the set of resonant points inside $I\times]0,R]$ is finite; therefore, one can always find a smaller, non-empty interval $\tilde{I} \subset I$ such that there is *no* resonant point at all in $\tilde{I}\times]0,R]$.

We extract the periods τ_k from $\mathcal{L}_1 := \mathcal{L}(\tilde{I}) \cap (\tilde{I}\times]0,R])$ inductively, as follows.

1. Consider a point $(\lambda_1,\tau_1) \in \mathcal{L}_1$ with "minimal height" τ_1 : $\forall(\lambda,\tau) \in \mathcal{L}_1, \tau_1 \leqslant \tau$.

2. By the non-resonance assumption, the connected component of (λ_1,τ_1) in $\mathcal{L}_1$ is the graph of a smooth function of the interval $\tilde{I}$. We denote this function by $\lambda \mapsto \tau_1(\lambda)$.

3. Consider the set

$$\mathcal{L}_2 := \mathcal{L}_1 \setminus \{(\lambda, j\tau_1(\lambda)); \quad \lambda \in \tilde{I}, j \in \mathbb{Z}^*\}.$$

   Again by the non-resonance assumption, $\mathcal{L}_1$ remains a union of non-intersecting smooth graphs.

4. If $\mathcal{L}_1$ is empty, then $N = 1$. Otherwise, start again by replacing $\mathcal{L}_0$ by $\mathcal{L}_1$, and so on. If $\mathcal{L}_k$ is empty, then $N = k - 1$.

$\square$

Remark 4.3 If we disregard symmetry issues, our assumption on the resonant set is quite weak. For instance, one can allow the crossing of two periods to be flat (all derivatives are equal at a point λ), simply because we put ourselves in a region with no crossing at all. However, it is easy to prove Step 2 with even weaker assumptions. For instance, it may work even if there are some open intervals of values of λ which admits resonant pairs. It would be interesting to know whether Step 1 could hold in this case as well. It would then involve sub-principal terms in the Bohr-Sommerfeld expansion. $\triangle$

4.2 Singular fibres

As we already mentioned, the following result comes for free.

Theorem 4.4 *Let Assumption $\mathcal{A}(P, \mathcal{J}, I)$ hold, and let $E_0 \in I$ be a non-degenerate critical value of p. Assume also that $p^{-1}(E_0)$ contains only one critical point. Then from the knowledge of $\Sigma(P, \mathcal{J}, I) + \mathcal{O}(\hbar^2)$ one can determine the topology of the singular foliation induced by p, in a saturated neighbourhood of $p^{-1}(E_0)$.*

Proof . Under these assumptions, the topology of the singular foliation induced by p in a saturated neighbourhood of $p^{-1}(E_0)$ is known to be completely characterised by the type of the singularity [8, 24], which is determined by Theorem 3.1. For the convenience of the reader, we briefly recall the two possible cases.

1. *The elliptic case.* — The singular fibre $p^{-1}(E_0)$ is just a point and the foliation is homeomorphic to the one given by the Hamiltonian $H(x, \xi) = x^2 + \xi^2$.

2. *The hyperbolic case.* — The singular fibre is a circle with a transversal self-intersection (the figure eight). It separates a saturated neighbourhood into three connected parts : two on one side, and one on the other side. It is homeomorphic to the foliation given by the Hamiltonian $H(x, \xi) = \xi^2 + x^4 - x^2$, in a neighbourhood of $H^{-1}(0)$.

□

4.3 Global topology

We say that a hamiltonian system p on the symplectic 2-manifold M is topologically equivalent to the hamiltonian system $\tilde{p}$ on $\tilde{M}$ if there is a homeomorphism $\varphi : M \to \tilde{M}$ such that

$$p = \tilde{p} \circ \varphi.$$

Notice that this implies that φ respects the foliation, fibre by fibre. In particular, p and $\tilde{p}$ have the same set of regular values and the same set of critical values. If I is an open interval, then two hamiltonian systems p and $\tilde{p}$ are called topologically equivalent over I when they are topologically equivalent when restricted to the symplectic manifolds $p^{-1}(I)$, $(\tilde{p})^{-1}(I)$.

We call the *topological type* of a hamiltonian system the equivalence class of topologically equivalent systems.

Theorem 4.5 *Let Assumption $\mathcal{A}(P, \mathcal{J}, I)$ hold, and assume that p has only nondegenerate critical values in some neighbourhood of I, such that any two critical points with the same singularity type cannot have the same image by p. Let $c_1 < \cdots < c_n$ be the critical values of p in I. Suppose that in each interval (c_i, c_{i+1}), $i = 1, \ldots, n-1$, there exists a non-empty subinterval I_i such that the set of resonant points in $I_i \times \mathbb{R}$ is discrete in $\mathbb{R}^2$. Then the knowledge of $\Sigma(P, \mathcal{J}, I) + \mathcal{O}(\hbar^2)$ determines the topological type of the hamiltonian system p over I.*

Proof. Upon a possible enlargement of I, one may assume that $I = (E_0, E_1)$ for regular values E_0, E_1. Using symplectic cutting [13] or surgery [24], one may replace the phase space $\mathbb{R}^4$ by a compact symplectic manifold where $p^{-1}(I)$ is embedded. Then we apply the result of [8] that says that the topological type of p on M is determined by its Reeb graph : the set of leaves of the foliation, as a topological 1-complex. This graph is characterised by the relative positions of critical values, and the number of fibres between two consecutive critical values. The former is determined by the spectrum in I

thanks to Theorem 3.1, while the latter is determined for each $i = 1, \ldots, n-1$ by the spectrum in I_i, thanks to Theorem 4.2. This give the topological type of p, up to some homeomorphism of the Reeb graph itself. But since we know the precise values of p at singularities, we can in fact assume that this homeomorphism is the identity. $\square$

5 Symplectic geometry

The tools we've used so far give us the *periods* of the classical hamiltonian system, which is of course much more than a mere topological information. We show here that it is indeed sufficient to recover the full dynamics of the systems.

We say that a hamiltonian system p on the symplectic 2-manifold M is symplectically equivalent to the hamiltonian system $\tilde{p}$ on $\tilde{M}$ if there is a smooth symplectomorphism $\varphi : M \to \tilde{M}$ such that

$$p = \tilde{p} \circ \varphi.$$

Thus, the dynamics of p on the levelset $\{p = E\}$ is transported via φ to the dynamics of $\tilde{p}$ on the levelset $\{\tilde{p} = E\}$.

We call the *symplectomorphism type* of a hamiltonian system the equivalence class of symplectically equivalent systems. As before, one may restrict this equivalence to an interval I of values of p and $\tilde{p}$.

Definition 5.1 *Let $(\lambda, t) \in I \times \mathbb{R}$ be a resonant point for p. Thus*

$$j\tau_k(\lambda) = j'\tau_{k'}(\lambda) = -t$$

for some $j, j', k \neq k'$. We say that this resonance is weakly transversal *if there exists an integer $n \in \mathbb{N}^*$ such that the n-th derivatives of the periods are not equal :*

$$j\tau_k^{(n)}(\lambda) \neq j'\tau_{k'}^{(n)}(\lambda).$$

Theorem 5.2 *Let Assumption $\mathcal{A}(P, \mathcal{J}, I)$ hold, and suppose that p has only nondegenerate critical values in some neighbourhood of I, such that any two critical points with the same singularity type cannot have the same image by p. Let $c_1 < \cdots < c_n$ be the critical values of p in I. Suppose that for each interval $J_i := (c_i, c_{i+1})$, $i = 1, \ldots, n-1$, the set of resonant points in $J_i \times \mathbb{R}$ is discrete. Finally assume that all such resonant points are weakly transversal.*

Then the knowledge of $\Sigma(P, \mathcal{J}, I) + \mathcal{O}(\hbar^2)$ determines the symplectic type of the hamiltonian system p over I.

Proof. We use the symplectic classification of [8, 17] using *weighted* Reeb graphs. Under our assumptions, the Reeb graph has vertices of degree 1 and 3. A vertex of degree 1, a *bout*, corresponds to an elliptic critical value, while a vertex of degree 3, called a *bifurcation point*, corresponds to a hyperbolic critical value. At a bifurcation point we can distinguish one particular edge, called the *trunk*, corresponding to the side of the figure 8 with only one connected component. The two other edges are called the *branches*. A weighted Reeb graph is a Reeb graph each of whose edges is associated with a positive real number, its *length*, and such that each of the two branches of each bifurcation point is associated with a formal Taylor series (*ie* a sequence of real numbers). The hypothesis of the theorem allow for determining the topological Reeb graph via Theorem 4.5. Thus, the next step of the proof is to show how the numbers that constitute the weighted Reeb graph can be recovered from the spectrum. The final step is to obtain the symplectic equivalence in the sense that we have just defined above.

The lengths. — Let $\mathcal{C}_k(J_i)$, for $k = 1, \ldots, N$, be the connected components of $p^{-1}(J_i)$. Let $K_{k,i} \in C^\infty(\mathcal{C}_k(J_i))$ be an action variable for the regular lagrangian fibration $p_{\restriction \mathcal{C}_k(J_i)}$; it is unique up to a sign and an additive constant. By definition the *length* of the edge corresponding to the set of leaves in $\mathcal{C}_k(J_i)$ is

$$\ell_{k,i} := \left| \lim_{c \to c_i} K_{k,i}(c) - \lim_{c \to c_{i+1}} K_{k,i}(c) \right|. \tag{12}$$

In a learned terminology, this is the Duistermaat-Heckman measure of J_i for the S^1-action defined by $K_{k,i}$, or, equivalently, it is the affine length of J_i endowed with its natural integral affine structure given by $p_{\restriction \mathcal{C}_k(J_i)}$.

It follows from the local models for elliptic and hyperbolic singularities that this length is always finite. This is obvious at elliptic singularities, where the action has the form $x^2 + \xi^2$. At a hyperbolic singularity m, one can introduce a foliation function q such that, in some local symplectic coordinates around m, $q = x\xi$, and $q > 0$ on the branches while $q < 0$ on the trunk. Then the Duistermaat-Heckman measure has the form

$$\begin{cases} d\mu_j(q) & = (\ln q + g_j(q))\, dq \quad \text{on each branch (j=1,2)} \\ d\mu(q) & = (2 \ln |q| + g(q))\, dq \quad \text{on the trunk,} \end{cases} \tag{13}$$

with some smooth functions g, g_1, g_2 satisfying

$$\forall p, \qquad g^{(p)}(0) = g_1^{(p)}(0) + g_2^{(p)}(0).$$

Under this form, the Taylor series of the functions g, g_1, g_2 at the origin are uniquely defined [17, 19].

Using the proof of Theorem 4.2, from the spectrum in I we can recover the periods $\tau_k(\lambda)$, $k = 1, \ldots, N$, for λ in any interval in J_i where the graphs of the periods τ_k don't cross. At a crossing the difficulty is to put the labels k correctly, so that the connected components $\mathcal{C}_k(\lambda)$ remain in the same $\mathcal{C}_k(J_i)$ when λ varies. This can be overcome precisely thanks to the weak resonant assumption at each crossing, because each τ_k is C^∞ in J_i. This was the main issue. Now, fixing a point $\lambda_i \in J_i$, the action variable $K_{k,i}$ can be computed by the formula

$$K_{k,i}(\lambda) := \int_{\lambda_i}^{\lambda} \tau_k(\lambda) d\lambda, \quad \lambda \in J_i.$$

This gives the length of $\mathcal{C}_k(J_i)$ via equation (12).

The Taylor series at the bifurcation points. — By definition, the sequences of numbers associated with a bifurcation point in the Reeb graph are the Taylor series of the functions g_1, g_2 (defined in equation (13)) at the origin.

Let us show how to recover the Taylor series of g from the spectrum. The procedure is completely analogous for g_1 and g_2.

Thus, we consider a hyperbolic critical value c_{i+1}. We want to express the Duistermaat-Heckman measure on the trunk in terms of the principal symbol p. By a theorem of Colin de Verdière and Vey [4], there exist local symplectic coordinates (x, ξ) at the hyperbolic point, and a smooth, locally invertible function $f : (\mathbb{R}, c_{i+1}) \to (\mathbb{R}, 0)$ such that

$$f(p) = x\xi = q.$$

For notational purposes, one may assume that $f'(c_{i+1}) > 0$, which amounts to say that the trunk is sent by p to $\lambda < c_{i+1}$. Then from (13), for λ close to c_{i+1}, $\lambda < c_{i+1}$,

$$d\mu(\lambda) = (2 \ln |f(\lambda)| + g \circ f(\lambda))\, f'(\lambda) d\lambda.$$

On the other hand if the connected component corresponding to the trunk is $\mathcal{C}_k(J_i)$, one has by definition of the Duistermaat-Heckman measure $d\mu(\lambda) = \tau_k(\lambda) d\lambda$. Therefore

$$\tau_k(\lambda) = f'(\lambda)\, (2 \ln |f(\lambda)| + g \circ f(\lambda)) = 2 f'(\lambda) \ln |\lambda - c_{i+1}| + h(\lambda),$$

for some smooth function h at $\lambda = c_{i+1}$. There, using Taylor's formula, we have written $f(\lambda) = \alpha(\lambda - c_{i+1}) + (\lambda - c_{i+1})^2 \hat{f}(\lambda)$, with $\alpha > 0$ and $\hat{f}$ smooth at c_{i+1}, and hence

$$h(\lambda) = 2f'(\lambda) \ln \left| \alpha + (\lambda - c_{i+1})\hat{f}(\lambda) \right| + f'(\lambda) g \circ f(\lambda). \tag{14}$$

This shows that h is smooth for λ close to c_{i+1}.

It is easy to see that any smooth function ϕ in a neighbourhood of the origin such that $\phi(t) \ln t$ extends to a smooth function at $t = 0$ must be flat. Hence the knowledge of $\tau_k(\lambda)$ for $\lambda < c_{i+1}$ completely determines the Taylor series of $f'(\lambda)$ (and hence $f(\lambda)$) at $\lambda = c_{i+1}$.

Then one can recover the Taylor series of h using

$$h(\lambda) = \tau_k(\lambda) - 2f'(\lambda) \ln |\lambda - c_{i+1}|, \quad \forall \lambda < c_{i+1}$$

Finally, from (14) and the fact that f is locally invertible, one can recover the Taylor series of g at the origin.

Symplectic equivalence. — We have proven that the weighted Reeb graph is determined by the spectrum. By Toulet's classification [8, 17], if two such systems (M, p) and $(M, \tilde{p})$ have the same weighted Reeb graph, there exists a symplectomorphism $\varphi : M \to \tilde{M}$ such that p and $\tilde{p} \circ \varphi$ define the same singular foliation on M (φ indices a homeomorphism of the leaf space, fixing the vertices). If we assume that the operators P and $\tilde{P}$ have the same spectrum (modulo $\hbar^2$) and fulfil the requirements of the theorem, then we also know that p and $\tilde{p} \circ \varphi$ share the same set of critical values c_i. The fact that p and $\tilde{p} \circ \varphi$ define the same foliation implies that for each connected component $\mathcal{C}_k(J_i)$, there exists a smooth, invertible function $f : J_i \to J_i$ such that

$$p = f \circ \tilde{p} \circ \varphi \qquad \text{on } \mathcal{C}_k(J_i). \tag{15}$$

Since the singular fibres at the ends of $\mathcal{C}_k(J_i)$ are fixed by φ, f must be increasing, and thus extends to a homeomorphism of $\overline{J_i}$.

As we already saw, the spectrum also determines the periods at a given energy $E = \lambda$. Hence for $\lambda \in J_i$, $\tau_k(\lambda) = \tilde{\tau}_k(\lambda)$. Since τ_k is integrable at c_{i+1}, we can define action integrals for $\lambda < c_{i+1}$ as :

$$K_{k,i}(\lambda) := \int_{c_{i+1}}^{\lambda} \tau_k(\lambda) d\lambda, \qquad \tilde{K}_{k,i}(\lambda) := \int_{c_{i+1}}^{\lambda} \tilde{\tau}_k(\lambda) d\lambda.$$

We have $K_{k,i}(\lambda) = \tilde{K}_{k,i}(\lambda)$. On the other hand, the action is a symplectic invariant of the foliation. From (15) on can compute the action on the curve $\varphi(\mathcal{C}_k(f(\lambda))) = \tilde{\mathcal{C}}_k(\lambda) : K_{k,i}(f(\lambda)) = \tilde{K}_{k,i}(\lambda) + \text{const}$. Therefore

$$K_{k,i}(\lambda) = K_{k,i}(f(\lambda)).$$

Since τ_k does not vanish in J_i, $K_{k,i}$ is strictly monotonous on J_i. Therefore

$$f(\lambda) = \lambda, \qquad \forall \lambda \in J_i.$$

Thus $p = \tilde{p} \circ \varphi$ on each $\mathcal{C}_k$, and by continuity

$$p = \tilde{p} \circ \varphi \quad \text{ on } M.$$

This finishes the proof of the theorem. □

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